Foundations of Nonlinear Time Series Analysis

AMS Spring 2024 Central Sectional Conference

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Outline

What Is Next

History

Nonlinear time series analysis

- Traces back to Henri Poincaré's three-body problem in the late 1800s.
  - Refers to the challenge of accurately predicting the future positions and motions of three celestial bodies.
  - Poincaré laid the foundations of chaos theory.
- In the 1960s, Edward Lorenz formulated a system of equations.
  - Highlighted the sensitivity to initial conditions, a hallmark of chaotic systems.
  - Guided researchers towards nonlinear dynamics.

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- Time series
  - Think of pictures in a video– the video is a time series of pictures.
- Trajectory: the path created when we plot data points; shows us how a system changes.
- Deterministic vs. Nondeterministic systems
  - Deterministic systems' future behavior can be precisely predicted given their initial state.
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State Space Reconstruction

- Abstract representation of a dynamic system's complete condition.
  - A multidimensional space where each dimension corresponds to a variable that somehow describes the system. Consider how temperature describes the overall weather.
  - A *state* represents a snapshot of the system. Analogous to a data point in a time series.
- But why use this technique?
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Delay Coordinate Embedding

- Reconstructs a state space from a single time series.
  - More than one dimension from a scalar time series– how is that possible?
- Consider a scalar measurement x, say temperature. We can construct an m-dimensional vector $\overrightarrow{R}(t)$ from m time-delayed measurements $x(t)$, such that
  $$\overrightarrow{R}(t) = [x(t), x(t-\tau), x(t-2\tau), ..., x(t-(m-1)\tau)]$$
  where t is the time of measurement and τ is the chosen time delay. The time-delay variable τ represents the intervals of these measurements.
- How does this translate to practical work?

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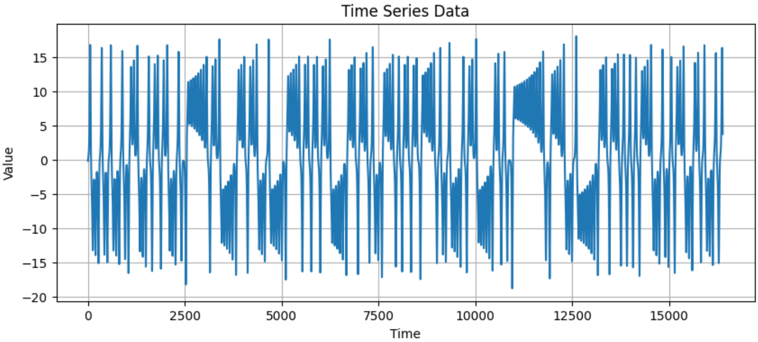
Delay Coordinate Embedding Cont.

Example to visualize

Consider the following time series



Delay Coordinate Embedding Cont.

Example to visualize

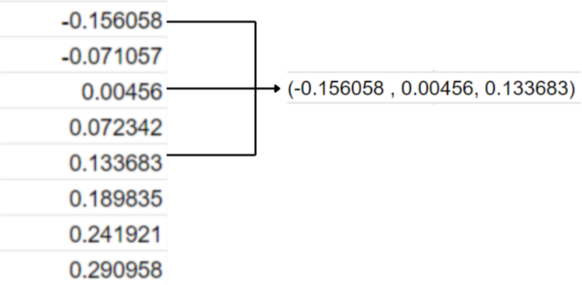
Here are the first eight data points

-0.156058
-0.071057
0.00456
0.072342
0.133683
0.189835
0.241921
0.290958

Delay Coordinate Embedding Cont.

Example to visualize

Let's embed some points!



Notice the time interval τ is two. Choosing τ is also a difficult task that is studied on its own– what value is too little vs. too much?

Delay Coordinate Embedding Cont.

Example to visualize

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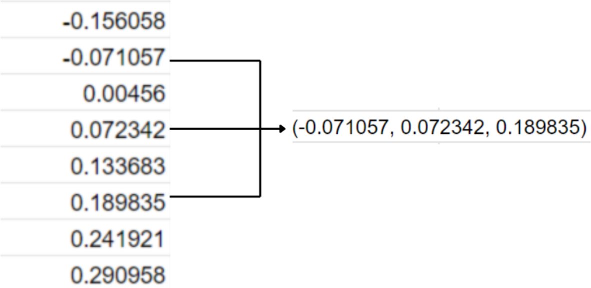

We can plot these points in a 3D space. When we do this for all data points, we can plot the trajectory.

What Is Next

Lyapunov Exponents, λ

- Russian mathematician Aleksandr Lyapunov.
- Quantifies the sensitivity of a dynamical system to its initial conditions.
- There is more than one exponent for a system– one for each variable. However, only one determines the overall behavior.
- $\lambda > 0$, signifies chaotic behavior within the system.
- Common method used: Rosenstein's algorithm

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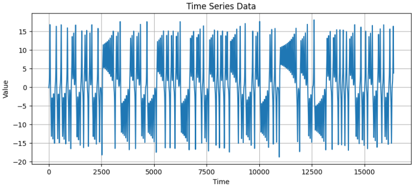

The Lyapunov exponent for this time series is 0.9056.

Attractors

- Mathematical constructs that provide insight into the long-term evolution of dynamic systems.
- Three kinds
  1. Fixed-point attractors represent rest.
  2. Periodic attractors represent repeating patterns or cycles, such as periodic orbits.
  3. Strange attractors represent complex, non-repeating attractors found in chaotic systems. Turbulence is a prime example of such behavior.
- Lorenz first noticed chaotic systems in the behavior of these three equations.

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Attractors Cont.

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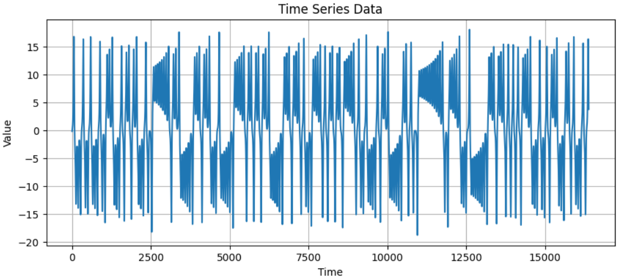
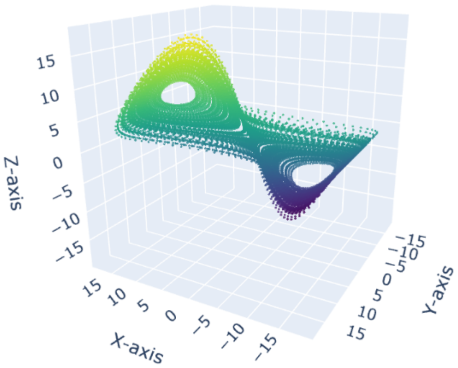
The time series produced by these equations looks like this



We have been looking at the Lorenz attractor's time series.

What does it look like?

Choosing $\tau = 10$, we get



Let's try other τ values to see their effect on reconstruction.
Click here!

What Is Next

Conclusion

Here is what we covered today:

1. Introduction
   - How NTSA came to be
   - Some important work in the field
2. Foundations
   - Basic terminology
   - State space reconstruction
   - Delay coordinate embedding
3. Characterization
   - Lyapunov exponents
   - Attractors

Overall, you should have a basic understanding of how nonlinear time series analysis works.

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Thank you for listening!

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